

Hypercontractivity and Applications for Stochastic Hamiltonian Systems ^{*}

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Abstract

The hypercontractivity is proved for the first time for the Markov semigroup associated to a class of finite/infinite dimensional stochastic Hamiltonian systems. Consequently, the Markov semigroup is exponentially convergent to the invariant probability measure in entropy (thus, also in L^2), and is compact for large time. These strengthen the hypocoercivity results derived in the literature. Since the log-Sobolev inequality is invalid for the associated Dirichlet form, we introduce a general result on the hypercontractivity using the Harnack inequality with power. The main results are illustrated by concrete examples.

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1 Introduction

In recent years, the hypocoercivity (i.e. L^2 -exponential convergence) has been intensively investigated for degenerate Fokker-Planck equations, see [3, 6, 7, 8, 10] and references within. A typical model is the following degenerate stochastic differential equation for $(X_t, Y_t)_{t \geq 0}$ on \mathbb{R}^{m+d} ($m, d \geq 1$):

$$(1.1) \quad \begin{cases} dX_t = (AX_t + BY_t) dt, \\ dY_t = Z(X_t, Y_t)dt + \sigma dW_t, \end{cases}$$

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where A, B and σ are matrices of orders $m \times m, m \times d$ and $d \times d$ respectively, W_t is the d -dimensional Brownian motion, and $Z : \mathbb{R}^{m+d} \rightarrow \mathbb{R}^d$ is continuous. This is a typical model of hypoelliptic diffusion processes provided σ is invertible and A, B satisfy the Kalman rank condition (see e.g. [21])

$$\text{Rank}[B, AB, \dots, A^{m-1}B] = m.$$

Moreover, when $Z(x, y) = -x + \nabla V(y)$ for some $V \in C^2(\mathbb{R}^d)$, the corresponding Kolmogorov-Fokker-Planck equation is well motivated from physics as kinetic Fokker-Planck equation with Hamiltonian potential V , see e.g. [14].

Let, for instance, $m = d, A = 0, \sigma = B = I$ and $Z(x, y) = -x - y$. In this case, the unique invariant probability measure μ of the associated Markov semigroup is the $2d$ -dimensional standard Gaussian distribution. Then the hypocoercivity derived in the literature is of type

$$(1.2) \quad \mu((P_t f)^2 + |\nabla P_t f|^2) \leq c e^{-\lambda t} \mu(f^2 + |\nabla f|^2), \quad t \geq 0, \mu(f) = 0,$$

for some constants $c, \lambda > 0$, where $\mu(f) := \int f d\mu$. Combining this with the gradient estimate $|\nabla P_1 f|^2 \leq C P_1 f^2$ (see [11, 21]) and using $P_1 f$ to replace f , we obtain the following exponential convergence in $L^2(\mu)$:

$$(1.3) \quad \mu((P_t f)^2) \leq c e^{-\lambda t} \mu(f^2), \quad t \geq 0, \mu(f) = 0.$$

In this paper, we aim to prove the following exponential convergence in entropy for the SDE (1.1):

$$(1.4) \quad \mu((P_t f) \log P_t f) \leq c e^{-\lambda t} \mu(f \log f), \quad f \geq 0, \mu(f) = 1, t \geq 0.$$

This inequality is stronger than (1.3) (see Proposition 2.3 below), and is more interesting from physics point of view since the entropy is an important quantity to measure the “disorder” of a thermodynamics system.

It should be indicated that the following type entropy inequality corresponding to (1.2) has been investigated in [3]:

$$\mu((P_t f) \log P_t f + (P_t f) |\nabla \log P_t f|^2) \leq c e^{-\lambda t} \mu(f \log f + f |\nabla \log f|^2), \quad f \geq 0, \mu(f) = 1, t \geq 0.$$

However, as we are not able to bound $\mu((P_1 f) |\nabla \log P_1 f|^2)$ by the entropy $\mu(f \log f)$, (1.4) does not follow from this inequality as (1.3) from (1.2).

It is well known that (1.4) follows from the hypercontractivity of P_t . According to [12], a Markov semigroup P_t is called hypercontractive with respect to its invariant probability measure μ , if $\|P_t\|_{2 \rightarrow 4} = 1$ for large $t > 0$, where $\|\cdot\|_{2 \rightarrow 4}$ is the operator norm from $L^2(\mu)$ to $L^4(\mu)$. Besides the exponential convergence in entropy (1.4), this property also implies the L^2 -compactness of P_t due to the existence of density, see the proof of Theorem 2.1 below for details. This stimulates us to investigate the hypercontractivity for the stochastic Hamilton system (1.1).

According to L. Gross (see e.g. [9]), the hypercontractivity follows from (in the symmetric case, is equivalent to) the log-Sobolev inequality

$$\mu(f^2 \log f^2) - \mu(f^2) \log \mu(f^2) \leq C \mathcal{E}(f, f), \quad f \in \mathcal{D}(\mathcal{E})$$

for some constant $C > 0$, where $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is the associated Dirichlet form. However, noting that in the present situation

$$\mathcal{E}(f, f) = \int_{\mathbb{R}^{m+d}} \sum_{i,j=1}^d (\sigma\sigma^*)_{ij} \{(\partial_{y_i} f)(\partial_{y_j} f)\}(x, y) \mu(dx, dy) = 0$$

provided f is independent of y , we see that the log-Sobolev inequality is invalid. So, to prove the hypercontractivity, we will use the dimension-free Harnack inequality initiated in [15].

It is known by [15] and has been used in many other papers that P_t is hyperbounded, i.e. $\|P_t\|_{2 \rightarrow p} < \infty$ for some $p > 2$, provided the Harnack inequality

$$(1.5) \quad (P_t f)^2(x) \leq (P_t f^2(y)) e^{C_t(x,y)}$$

holds for some measurable function C_t such that, for some $\varepsilon > 0$,

$$(1.6) \quad \int e^{(1+\varepsilon)C_t(x,y)} \mu(dx) \mu(dy) < \infty.$$

Although careful calculations are required, it is now standard to establish the Harnack inequality (1.5) using coupling by change of measures (cf. [18]), while (1.6) can be verified by properly choosing Lyapunov functions. So, the key point of the present study will be to deduce the hypercontractivity from the hyperboundedness. This is trivial in the symmetric setting, since as shown in [19] that for irreducible Dirichlet forms the log-Sobolev inequality is equivalent to the defective log-Sobolev inequality, so that the hyperboundedness is indeed equivalent to the hypercontractivity. However, this equivalence does not apply to the present setting, since the log-Sobolev inequality is not available as explained above. To overcome this difficulty, we will adopt an additional coupling argument which ensures that $\|P_t\|_{2 \rightarrow 4} < 2$ for large $t > 0$, and finally prove the hypercontractivity using a more general result introduced in Proposition 2.2 below.

We now introduce our main results in finite- and infinite-dimensional respectively.

(A) The finite dimensional case

In this part, we consider the equation (1.1) on \mathbb{R}^{m+d} . To prove the hypercontractivity, we need the following assumptions. Let $\|\cdot\|$ denote the operator norm for an matrix.

(A1) σ is invertible and $\text{Rank}[B, AB, \dots, A^{m-1}B] = m$.

(A2) $Z : \mathbb{R}^{m+d} \rightarrow \mathbb{R}^d$ is Lipschitz continuous.

(A3) There exist constants $\theta > 0$ and $r \in (-\|B\|^{-1}, \|B\|^{-1})$ such that

$$\begin{aligned} & \langle x - \bar{x} + rB(y - \bar{y}), A(x - \bar{x}) + B(y - \bar{y}) \rangle + \langle Z(x, y) - Z(\bar{x}, \bar{y}), y - \bar{y} + rB^*(x - \bar{x}) \rangle \\ & \leq -\theta(|x - \bar{x}|^2 + |y - \bar{y}|^2), \quad (x, y), (\bar{x}, \bar{y}) \in \mathbb{R}^{m+d}. \end{aligned}$$

Theorem 1.1. *Assume (A1), (A2) and (A3) and let P_t be the Markov semigroup associated to (1.1). Then:*

- (1) P_t has a unique invariant probability measure μ and $\mu(e^{\varepsilon|\cdot|^2}) < \infty$ for some $\varepsilon > 0$;
- (2) P_t is hypercontractive, i.e. $\|P_t\|_{2 \rightarrow 4} = 1$ for large $t > 0$;
- (3) P_t is compact in $L^2(\mu)$ for large $t > 0$, and there exist constants $c, \lambda > 0$ such that (1.4) holds.

(B) The infinite dimensional case

We consider the following infinite-dimensional version of (1.1) on $\mathbb{H} := \mathbb{H}_1 \times \mathbb{H}_2$ for two separable Hilbert spaces \mathbb{H}_1 and \mathbb{H}_2 :

$$(1.7) \quad \begin{cases} dX_t = (AX_t + BY_t - L_1X_t) dt, \\ dY_t = \{Z(X_t, Y_t) - L_2Y_t\}dt + \sigma dW_t, \end{cases}$$

where W_t is a cylindrical Brownian motion on \mathbb{H}_2 , i.e. $W_t = \sum_{i=1}^{\infty} B_t^i e_i$ for a sequence of independent one-dimensional Brownian motions $\{B_t^i\}_{i \geq 1}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an orthonormal basis $\{e_i\}_{i \geq 1}$ on \mathbb{H}_2 ; $A : \mathbb{H}_1 \rightarrow \mathbb{H}_1, B : \mathbb{H}_2 \rightarrow \mathbb{H}_1$ and $\sigma : \mathbb{H}_2 \rightarrow \mathbb{H}_2$ are bounded linear operators; $(L_i, \mathcal{D}(L_i))$ is a positive definite self-adjoint operator on $\mathbb{H}_i, i = 1, 2$; and $Z : \mathbb{H} \rightarrow \mathbb{H}_2$ is measurable.

When \mathbb{H}_2 is infinite-dimensional, the noise σW_t does not exist on \mathbb{H}_2 . So, the unbounded operator L_2 plays a crucial role in the study of mild solutions (see [5]). The unbounded operator L_1 is the counterpart of L_2 for the first component process X_t , and the bounded operator A stands for a perturbation of L_1 , see (B3) below.

Let $\langle \cdot, \cdot \rangle, |\cdot|$ and $\|\cdot\|$ denote, respectively, the inner product, the norm and the operator norm on a Hilbert space. Moreover, for a linear operator $(L, \mathcal{D}(L))$ on a Hilbert space, and for $\lambda \in \mathbb{R}$, we write $L \geq \lambda$ if $\langle f, Lf \rangle \geq \lambda|f|^2$ holds for all $f \in \mathcal{D}(L)$.

To investigate the hypercontractivity of the associated Markov semigroup to (1.7), we will need the following assumptions.

(B1) σ is invertible, and L_2 has discrete spectrum with eigenbasis $\{e_i\}_{i \geq 1}$ and corresponding eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ including multiplicities. Assume $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$.

(B2) There exist two constants $K_1, K_2 > 0$ such that

$$|Z(x, y) - Z(\bar{x}, \bar{y})| \leq K_1|x - \bar{x}| + K_2|y - \bar{y}|, \quad (x, y), (\bar{x}, \bar{y}) \in \mathbb{H}.$$

(B3) $L_1 - A \geq \lambda_1 - \delta$ for some constant $\delta \geq 0$, $BL_2 = L_1B$, $AL_1 = L_1A$, and for any $t > 0$, the operator

$$Q_t := \int_0^t e^{sA} B B^* e^{sA^*} ds$$

is invertible on \mathbb{H}_1 .

It is well known that (B1) and (B2) imply the existence and uniqueness of mild solutions to (1.7), see [5]. Let P_t be the Markov semigroup associated to (1.7).

Theorem 1.2. *Assume (B1), (B2) and (B3). If*

$$(1.8) \quad \lambda' := \frac{1}{2} \left(\delta + K_2 + \sqrt{(K_2 - \delta)^2 + 4K_1 \|B\|} \right) < \lambda_1,$$

then the assertions in Theorem 1.1 hold.

The remainder of the paper is organized as follows. In Section 2, we present a general result on the hypercontractivity using the dimension-free Harnack inequality, which is then used in Sections 3 and 4 to prove Theorems 1.1 and 1.2 respectively. Some concrete Examples including the kinetic Fokker-Planck equations are presented in Section 4.

2 Hypercontractivity using Harnack inequality

In this section, we aim to introduce a general result on the hypercontractivity using Harnack inequality. The basic idea of the study goes back to [15] for elliptic diffusion semigroups on manifolds, see also [2] for a recent study of functional SDEs.

Let (E, \mathcal{B}, μ) be a probability space, and let P_t be a Markov semigroup on $\mathcal{B}_b(E)$ such that μ is P_t -invariant. Recall that a process (X_t, Y_t) on $E \times E$ is called a coupling associated to the semigroup P_t , if

$$(P_t f)(X_0) = \mathbb{E}(f(X_t)|X_0), \quad (P_t f)(Y_0) = \mathbb{E}(f(Y_t)|Y_0), \quad f \in \mathcal{B}_b(E), t \geq 0.$$

Theorem 2.1. *Assume that the following three conditions hold for some measurable functions $\rho : E \times E \rightarrow (0, \infty)$ and $\phi : [0, \infty) \rightarrow (0, \infty)$ with $\lim_{t \rightarrow \infty} \phi(t) = 0$:*

(i) *There exists two constants $t_0, c_0 > 0$ such that*

$$(P_{t_0} f(\xi))^2 \leq (P_{t_0} f^2(\eta)) e^{c_0 \rho(\xi, \eta)^2}, \quad f \in \mathcal{B}_b(E), \xi, \eta \in E;$$

(ii) *For any $(X_0, Y_0) \in E \times E$, there exists a coupling (X_t, Y_t) associated to P_t such that*

$$\rho(X_t, Y_t) \leq \phi(t) \rho(X_0, Y_0), \quad t \geq 0;$$

(iii) *There exists $\varepsilon > 0$ such that $(\mu \times \mu)(e^{\varepsilon \rho^2}) < \infty$.*

Then μ is the unique invariant probability measure of P_t , P_t is hypercontractive and compact in $L^2(\mu)$ for large $t > 0$. Consequently, (1.3) and (1.4) hold for some constants $c, \lambda > 0$.

To prove this result, we introduce the following propositions.

Proposition 2.2. *Let P be a bounded linear operator on $L^2(\mu)$ such that $P1 = 1$ and μ is P -invariant, i.e. $\mu(Pf) = \mu(f)$ for $f \in L^2(\mu)$. If $\|P\|_{2 \rightarrow 4}^4 < 2$, then:*

$$(1) \|P - \mu\|_2 := \sup_{\mu(f^2) \leq 1} \sqrt{\mu((Pf - \mu(f))^2)} < 1;$$

$$(2) \text{ There exists } n \geq 1 \text{ such that } \|P^n\|_{2 \rightarrow 4} = 1.$$

Proof. (1) The proof of the first assertion is essentially due to [16] where symmetric Markov operators are considered. Here, we show that the argument works also for the present situation.

Let $\delta(P) := \|P\|_{2 \rightarrow 4}^4 < 2$. For any $f \in L^2(\mu)$ with $\mu(f^2) = 1$ and $\mu(f) = 0$, we intend to prove

$$(2.1) \quad \mu((Pf)^2) \leq \sup_{\varepsilon \in (0,1)} \frac{\sqrt{8\varepsilon^2 + \delta(P)} - 3\varepsilon}{1 - \varepsilon}.$$

Without loss of generality, we assume $\mu((Pf)^3) \geq 0$, otherwise it suffices to replace f by $-f$. To prove this inequality, for any $\varepsilon \in (0, 1)$ we let $g_\varepsilon = \sqrt{\varepsilon} + \sqrt{1 - \varepsilon}f$. Then $\mu(g_\varepsilon^2) = 1$. Since $P1 = 1$, $\mu(Pf) = \mu(f) = 0$, $\mu((Pf)^3) \geq 0$, $\mu(g_\varepsilon^2) = 1$ and $\mu((Pf)^4) \geq \mu((Pf)^2)^2$, we have

$$\begin{aligned} \delta(P) &\geq \mu((Pg_\varepsilon)^4) \\ &= \varepsilon^2 + (1 - \varepsilon)^2 \mu((Pf)^4) + 6\varepsilon(1 - \varepsilon) \mu((Pf)^2) + 4\varepsilon^{\frac{3}{2}} \sqrt{1 - \varepsilon} \mu(Pf) + 4\sqrt{\varepsilon}(1 - \varepsilon)^{\frac{3}{2}} \mu((Pf)^3) \\ &\geq (1 - \varepsilon)^2 \mu((Pf)^2)^2 + 6\varepsilon(1 - \varepsilon) \mu((Pf)^2) + \varepsilon^2. \end{aligned}$$

This implies (2.1). Therefore,

$$\|P - \mu\|_2^2 \leq \inf_{\varepsilon \in (0,1)} \frac{\sqrt{8\varepsilon^2 + \delta(P)} - 3\varepsilon}{1 - \varepsilon} < 1$$

provided $\delta(P) < 2$, see pages 2632-2633 in [16] for the calculations of the inf.

(2) The proof of the second assertion is standard in the literature of continuous time Markov semigroups. For $f \in L^2(\mu)$ with $\mu(f^2) = 1$, let $\hat{f} = f - \mu(f)$. We have $\mu(P^k \hat{f}) = 0$, $k \geq 1$, so that by (1),

$$\mu((P^k \hat{f})^2) \leq \varepsilon^{2k} \mu(\hat{f}^2), \quad k \geq 1$$

holds for $\varepsilon := \|P - \mu\|_2 < 1$. Then for any $m \geq 1$,

$$\begin{aligned} \mu((P^{m+1} f)^4) &= \mu(f)^4 + 4\mu(f) \mu((P^{m+1} \hat{f})^3) + 6\mu(f)^2 \mu((P^{m+1} \hat{f})^2) + \mu((P^{m+1} \hat{f})^4) \\ &\leq \mu(f)^4 + 4\|P\|_{2 \rightarrow 3}^3 |\mu(f)| \mu((P^m \hat{f})^2)^{\frac{3}{2}} + 6\mu(f)^2 \mu((P^{m+1} \hat{f})^2) + \|P\|_{2 \rightarrow 4}^4 \mu((P^m \hat{f})^2)^2 \\ &\leq \mu(f)^4 + 4\|P\|_{2 \rightarrow 3}^3 \varepsilon^{3m} |\mu(f)| \mu(\hat{f}^2)^{\frac{3}{2}} + 6\varepsilon^{2(m+1)} \mu(f)^2 \mu(\hat{f}^2) + \|P\|_{2 \rightarrow 4}^4 \varepsilon^{4m} \mu(\hat{f}^2)^2. \end{aligned}$$

Since $\varepsilon \in (0, 1)$, $\|P\|_{2 \rightarrow 3} \leq \|P\|_{2 \rightarrow 4} < \infty$, and

$$|\mu(f)| \mu(\hat{f}^2)^{\frac{3}{2}} \leq \mu(f)^2 \mu(\hat{f}^2) + \mu(\hat{f}^2)^2,$$

this implies that for large enough $m \geq 1$,

$$\mu((P^{m+1} f)^4) \leq \mu(f)^4 + 2\mu(f)^2 \mu(\hat{f}^2) + \mu(\hat{f}^2)^2 = \mu(f^2)^2 = 1.$$

Thus, $\|P^n\|_{2 \rightarrow 4} \leq 1$ holds for large enough $n \geq 1$. □

Next, we present a result on exponential convergence in entropy implied by the hypercontractivity, which is well-known in the literature of symmetric Markov semigroups, but is new for general hypercontractive operators.

Proposition 2.3. *Let P be a positivity-preserving linear operator on $L^1(\mu)$ such that μ is P -invariant and $\|P\|_{p \rightarrow q} \leq 1$ holds for some constants $q > p > 1$. Then*

$$(2.2) \quad \mu((Pf) \log Pf) \leq \frac{(p-1)q}{p(q-1)} \mu(f \log f), \quad f \geq 0, \mu(f) = 1.$$

Consequently,

$$(2.3) \quad \mu((Pf - \mu(f))^2) \leq \frac{(p-1)q}{p(q-1)} \mu((f - \mu(f))^2), \quad f \in L^2(\mu).$$

Proof. For f , by applying (2.2) to $f_s := \frac{1+sf}{1+s\mu(f)}$, multiplying with s^{-2} and letting $s \rightarrow 0$, we derive (2.3). So, it suffices to prove (2.2). For any $\varepsilon \in (0, p-1)$, let

$$r = \frac{p-1-\varepsilon}{(1+\varepsilon)(p-1)}, \quad \delta(\varepsilon) = \frac{p(q-1)\varepsilon}{(p-1-\varepsilon)q + \varepsilon p}.$$

Then

$$\frac{1}{1+\varepsilon} = r + \frac{1-r}{p}, \quad \frac{1}{1+\delta(\varepsilon)} = r + \frac{1-r}{q}.$$

Since $\|P\|_{1 \rightarrow 1} = 1$ and $\|P\|_{p \rightarrow q} \leq 1$, by Riesz-Thorin's interpolation Theorem, we have $\|P\|_{1+\varepsilon \rightarrow 1+\delta(\varepsilon)} \leq 1$. So, for any $f \geq 1$ with $\mu(f) = 1$,

$$\int (Pf^{\frac{1}{1+\varepsilon}})^{1+\delta(\varepsilon)} d\mu \leq 1, \quad \varepsilon \in (0, p-1).$$

As the equality holds for $\varepsilon = 0$, we have

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int (Pf^{\frac{1}{1+\varepsilon}})^{1+\delta(\varepsilon)} d\mu \leq 0.$$

This implies (2.2). □

Proof of Theorem 2.1. (a) According to [20, Proposition 3.1], (i) implies that μ is the unique invariant probability measure of P_{t_0} , and P_{t_0} has a density with respect to μ . Thus, by [22, Theorem 2.3], P_{t_0+t} is compact in $L^2(\mu)$ for $t > 0$ such that $\|P_t\|_{2 \rightarrow 4} < \infty$. Therefore, according to Propositions 2.2 and 2.3, it suffices to prove $\|P_t\|_{2 \rightarrow 4}^4 < 2$ for large enough $t > 0$.

(b) By (i) and (ii) for $(X_0, Y_0) = (\xi, \eta)$,

$$(P_{t+t_0}f(\xi))^2 \leq \mathbb{E}(P_{t_0}f(X_t))^2 \leq \mathbb{E}\left[(P_{t_0}f^2(Y_t))e^{c_0\rho(X_t, Y_t)^2}\right] \leq (P_{t_0+t}f^2(\eta))e^{c_0\phi(t)^2\rho(\xi, \eta)^2}$$

holds for all $t \geq 0$, $f \in \mathcal{B}_b(E)$, and $\xi, \eta \in E$. Then for $\mu(f^2) \leq 1$, we have

$$(P_{t_0+t}f(\xi))^2 \int_E e^{-c_0\phi(t)^2\rho(\xi,\eta)^2} \mu(d\eta) \leq \int_M P_t f^2(\eta) \mu(d\eta) = \mu(f^2) \leq 1.$$

So,

$$(P_{t_0+t}f(\xi))^4 \leq \frac{1}{\left(\int_E \exp[-c_0\phi(t)^2\rho(\xi,\eta)^2] \mu(d\eta)\right)^2}, \quad \mu(f^2) \leq 1,$$

and by Jensen's inequality,

$$\begin{aligned} \sup_{\mu(f^2) \leq 1} \int_E (P_{t_0+t}f(\xi))^4 \mu(d\xi) &\leq \int_E \frac{\mu(d\xi)}{\left(\int_E \exp[-c_0\phi(t)^2\rho(\xi,\eta)^2] \mu(d\eta)\right)^2} \\ &\leq \int_E \left(\int_E e^{c_0\phi(t)^2\rho(\xi,\eta)^2} \mu(d\eta)\right)^2 \mu(d\xi) \leq \int_{E \times E} e^{2c_0\phi(t)^2\rho(\xi,\eta)^2} \mu(d\xi) \mu(d\eta) < 2 \end{aligned}$$

for large enough $t > 0$, since $\lim_{t \rightarrow \infty} \phi(t) = 0$ and (iii) imply

$$\lim_{t \rightarrow \infty} \int_{E \times E} e^{2c_0\phi(t)^2\rho(\xi,\eta)^2} \mu(d\xi) \mu(d\eta) = 1.$$

Therefore, $\|P_t\|_{2 \rightarrow 4}^4 < 2$ for large enough $t > 0$. □

3 Proof of Theorem 1.1

According to Theorem 2.1 and Proposition 2.3, it suffices to prove the following three lemmas corresponding to conditions (i)-(iii) respectively. The first lemma provides the desired Harnack inequality which was also investigated in [11, 21] for stochastic Hamiltonian systems. However, the inequality presented in [11] (see Corollary 4.2 therein) contains a worse exponential term, while the assumption (H) in [21] does not hold if Z is not second order differentiable. So, we present here a simple proof of the Harnack inequality using coupling by change of measures. See [18, Chapter 1] for more results on the coupling by change measures and applications.

Lemma 3.1. *Assume (A1) and (A2). For any $t_0 > 0$ there exists a constant $c_0 > 0$ such that*

$$(P_{t_0}f)^2(\xi) \leq (P_{t_0}f^2(\eta))e^{c_0|\xi-\eta|^2}, \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}), \xi, \eta \in \mathbb{R}^{m+d}.$$

Proof. Let (X_t, Y_t) solve the equation (1.1) with $(X_0, Y_0) = \eta \in \mathbb{R}^{m+d}$, and let (\bar{X}_t, \bar{Y}_t) solve the following equation with $(X_0, Y_0) = \xi \in \mathbb{R}^{m+d}$:

$$(3.1) \quad \begin{cases} d\bar{X}_t = (A\bar{X}_t + B\bar{Y}_t) dt, \\ d\bar{Y}_t = \left\{ Z(X_t, Y_t) + \frac{Y_0 - \bar{Y}_0}{t_0} + \frac{d}{dt}(t(t_0 - t)B^*e^{(t_0-t)A^*}b) \right\} dt + \sigma dW_t, \end{cases}$$

where $b \in \mathbb{R}^m$ is to be determined such that $(X_{t_0}, Y_{t_0}) = (\bar{X}_{t_0}, \bar{Y}_{t_0})$. It is easy to see that

$$\begin{cases} \frac{d}{dt}(X_t - \bar{X}_t) = A(X_t - \bar{X}_t) + B(Y_t - \bar{Y}_t), \\ \frac{d}{dt}(Y_t - \bar{Y}_t) = \frac{1}{t_0}(\bar{Y}_0 - Y_0) - \frac{d}{dt}\{t(t_0 - t)B^*e^{(t_0-t)A^*}b\}. \end{cases}$$

So,

$$(3.2) \quad Y_t - \bar{Y}_t = \frac{t_0 - t}{t_0}(Y_0 - \bar{Y}_0) - t(t_0 - t)B^*e^{(t_0-t)A^*}b,$$

and thus,

$$\begin{aligned} (3.3) \quad X_t - \bar{X}_t &= e^{At}(X_0 - \bar{X}_0) + \int_0^t e^{A(t-s)}B(Y_s - \bar{Y}_s)ds \\ &= e^{At}(X_0 - \bar{X}_0) + \left(\int_0^t e^{A(t-s)} \frac{t_0 - s}{t_0} ds \right) B(Y_0 - \bar{Y}_0) \\ &\quad - \left(\int_0^t s(t_0 - s) e^{A(t-s)} BB^* e^{(t_0-s)A^*} ds \right) b. \end{aligned}$$

Obviously, we have $Y_{t_0} = \bar{Y}_{t_0}$ and $X_{t_0} = \bar{X}_{t_0}$ provided

$$(3.4) \quad b = Q_{t_0}^{-1} \left\{ e^{t_0 A}(X_0 - \bar{X}_0) + \left(\int_0^{t_0} \frac{t_0 - s}{t_0} e^{A(t_0-s)} ds \right) B(Y_0 - \bar{Y}_0) \right\},$$

where, according to [13, §3], the rank condition (A1) implies that

$$Q_{t_0} := \int_0^{t_0} s(t_0 - s) e^{A(t_0-s)} BB^* e^{(t_0-s)A^*} ds$$

is invertible on \mathbb{R}^m , see (1) in the proof of [21, Theorem 4.2] for details.

Now, let b be in (3.4). We have $(X_{t_0}, Y_{t_0}) = (\bar{X}_{t_0}, \bar{Y}_{t_0})$. In order to establish the Harnack inequality, let

$$\psi_t = Z(X_t, Y_t) - Z(\bar{X}_t, \bar{Y}_t) + \frac{1}{t_0}(Y_0 - \bar{Y}_0) + \frac{d}{dt}\{t(t_0 - t)B^*e^{(t_0-t)A^*}b\}, \quad t \in [0, t_0].$$

Since Z is Lipschitz continuous, (4.10), (3.3) and (3.4) imply

$$(3.5) \quad |\psi_t|^2 \leq c_1(|X_0 - \bar{X}_0|^2 + |Y_0 - \bar{Y}_0|^2) = c_1|\xi - \eta|^2, \quad t \in [0, t_0]$$

for some constant $c_1 > 0$. Moreover, according to the definition of ψ , (3.1) can be reformulated as

$$\begin{cases} d\bar{X}_t = (A\bar{X}_t + B\bar{Y}_t) dt, \\ d\bar{Y}_t = Z(\bar{X}_t, \bar{Y}_t)dt + \sigma d\bar{W}_t, \end{cases}$$

where, by (3.5) and Girsanov's theorem,

$$\bar{W}_t := W_t + \sigma^{-1} \int_0^t \psi_s ds, \quad t \in [0, t_0]$$

is a d -dimensional Brownian motion under the probability measure $d\mathbb{Q} := Rd\mathbb{P}$ for

$$(3.6) \quad R := \exp \left[- \int_0^{t_0} \langle \sigma^{-1} \psi_t, dW_t \rangle - \frac{1}{2} \int_0^{t_0} |\sigma^{-1} \psi_t|^2 dt \right].$$

Therefore, by the weak uniqueness of the equation (1.1), and noting that $(X_{t_0}, Y_{t_0}) = (\bar{X}_{t_0}, \bar{Y}_{t_0})$, we have

$$\begin{aligned} (P_{t_0} f(\xi))^2 &= (\mathbb{E}[Rf(\bar{X}_{t_0}, \bar{Y}_{t_0})])^2 = (\mathbb{E}[Rf(X_{t_0}, Y_{t_0})])^2 \\ &\leq (\mathbb{E}R^2) \mathbb{E}f^2(X_{t_0}, Y_{t_0}) = (P_{t_0} f^2(\eta)) \mathbb{E}R^2. \end{aligned}$$

Then the proof is finished by noting that (3.5) and (3.6) imply $\mathbb{E}R^2 \leq e^{c_0|\xi-\eta|^2}$ for some constant $c_0 > 0$. \square

Lemma 3.2. *Assume (A3). Then there exist two constants $c, \lambda > 0$ such that for any two solutions (X_t, Y_t) and $(\tilde{X}_t, \tilde{Y}_t)$ to (1.1), there holds*

$$|X_t - \tilde{X}_t|^2 + |Y_t - \tilde{Y}_t|^2 \leq ce^{-\lambda t}(|X_0 - \tilde{X}_0|^2 + |Y_0 - \tilde{Y}_0|^2), \quad t \geq 0.$$

Proof. We have

$$(3.7) \quad \begin{cases} \frac{d}{dt}(X_t - \tilde{X}_t) = A(X_t - \tilde{X}_t) + B(Y_t - \tilde{Y}_t), \\ \frac{d}{dt}(Y_t - \tilde{Y}_t) = (Z(X_t, Y_t) - Z(\tilde{X}_t, \tilde{Y}_t))dt. \end{cases}$$

Since $r \in (-\|B\|^{-1}, \|B\|^{-1})$, there exists a constant $C > 1$ such that

$$\begin{aligned} (3.8) \quad &\frac{1}{C}(|X_t - \tilde{X}_t|^2 + |Y_t - \tilde{Y}_t|^2) \\ &\leq \Phi_t := \frac{1}{2}|X_t - \tilde{X}_t|^2 + \frac{1}{2}|Y_t - \tilde{Y}_t|^2 + r\langle X_t - \tilde{X}_t, B(Y_t - \tilde{Y}_t) \rangle \\ &\leq C(|X_t - \tilde{X}_t|^2 + |Y_t - \tilde{Y}_t|^2), \quad t \geq 0. \end{aligned}$$

Combining this with (3.7) and (A3), we obtain

$$d\Phi_t \leq -\theta(|X_t - \tilde{X}_t|^2 + |Y_t - \tilde{Y}_t|^2) \leq -\frac{\theta}{C}\Phi_t dt.$$

Therefore, $\Phi_t \leq \Phi_0 e^{-\theta t/C}$. Combining this with (3.8) we finish the proof. \square

Lemma 3.3. *Assume (A3). Then P_t has an invariant probability measure μ such that $\mu(e^{\varepsilon|\cdot|^2}) < \infty$ holds for some constant $\varepsilon > 0$.*

Proof. Let (X_t, Y_t) solve (1.1) with $(X_0, Y_0) = 0 \in \mathbb{R}^{m+d}$. Noting that for any $\varepsilon > 0$ the function $e^{\varepsilon|\cdot|^2}$ has compact level sets, by a standard tightness argument it suffices to prove

$$(3.9) \quad \sup_{t \geq 0} \mathbb{E} e^{\varepsilon(|X_t|^2 + |Y_t|^2)} < \infty$$

for some constant $\varepsilon > 0$. Since $r \in (-\|B\|^{-1}, \|B\|^{-1})$, there exists a constant $C > 1$ such that

$$(3.10) \quad \begin{aligned} \frac{1}{C}(|X_t|^2 + |Y_t|^2) &\leq \Psi_t := \frac{1}{2}|X_t|^2 + \frac{1}{2}|Y_t|^2 + r\langle X_t, BY_t \rangle \\ &\leq C(|X_t|^2 + |Y_t|^2), \quad t \geq 0. \end{aligned}$$

Moreover, by (A3) with $(\bar{x}, \bar{y}) = 0$, we have

$$\langle x + rBy, Ax + By \rangle + \langle Z(x, y) - Z(0, 0), y + rB^*x \rangle \leq -\theta(|x|^2 + |y|^2), \quad (x, y) \in \mathbb{R}^{m+d}.$$

So,

$$\begin{aligned} &\langle x + rBy, Ax + By \rangle + \langle Z(x, y), y + rB^*x \rangle \\ &\leq |Z(0, 0)| \cdot |y + rB^*x| - \theta(|x|^2 + |y|^2) \leq c_1 - c_2(|x|^2 + |y|^2), \quad (x, y) \in \mathbb{R}^{m+d} \end{aligned}$$

holds for some constants $c_1, c_2 > 0$. Thus, by (1.1), Itô's formula and (3.10), there exist constants $c_3, c_4 > 0$ such that

$$\begin{aligned} d\Psi_t &\leq (c_3 - c_2(|X_t|^2 + |Y_t|^2))dt + \langle Y_t + rB^*X_t, \sigma dW_t \rangle \\ &\leq (c_3 - c_4\Psi_t)dt + \langle Y_t + rB^*X_t, \sigma dW_t \rangle. \end{aligned}$$

By Itô's formula, this implies that for any $\varepsilon > 0$,

$$de^{\varepsilon\Psi_t} \leq \varepsilon e^{\varepsilon\Psi_t} \left(c_3 - c_4\Psi_t + \frac{\varepsilon^2}{2} |\sigma^*(Y_t + rB^*X_t)|^2 \right) dt + dM_t$$

holds for some local martingale M_t . Noting that (3.10) implies $|\sigma^*(Y_t + rB^*X_t)|^2 \leq c_5\Psi_t$ for some constant $c_5 > 0$, by letting $\varepsilon = \frac{c_4}{c_5}$ we obtain

$$de^{\varepsilon\Psi_t} \leq \varepsilon e^{\varepsilon\Psi_t} \left(c_3 - \frac{1}{2}c_4\Psi_t \right) dt + dM_t \leq (c_6 - e^{\varepsilon\Psi_t})dt + dM_t$$

for some constant $c_6 \geq 1$. Since $e^{\varepsilon\Psi_0} = 1$, this implies

$$\mathbb{E}e^{\varepsilon\Psi_t} \leq c_6, \quad t \geq 0.$$

Therefore, according to (3.10), (3.9) holds for some constant $\varepsilon > 0$. □

4 Proof of Theorem 1.2

As in the proof of Theorem 1.1, we need to verify conditions in Theorem 2.1. Moreover, we also need to prove the existence of invariant probability measure. In the present case, Itô's formula is invalid for the distance function of the mild solution. So, we use the following formulation of the mild solution:

$$(4.1) \quad \begin{cases} X_t = e^{-(L_1 - A + \delta)t} X_0 + \int_0^t e^{-(L_1 - A + \delta)(t-s)} (\delta X_s + BY_s) ds, \\ Y_t = e^{-L_2 t} Y_0 + \int_0^t e^{-L_2(t-s)} Z(X_s, Y_s) ds + \xi_t, \end{cases}$$

where

$$\xi_t := \int_0^t e^{-L_2(t-s)} \sigma dW_s, \quad t \geq 0.$$

Due to (B1), for any $T > 0$, the process

$$M_t := \int_0^t e^{-L_2(T-s)} \sigma dW_s, \quad t \in [0, T]$$

is a square integrable martingale on \mathbb{H} with quadratic variation process

$$\langle M \rangle_t = \int_0^t \|e^{-L_2(T-s)} \sigma\|_{HS}^2 ds \leq \|\sigma\|^2 \sum_{i=1}^{\infty} \frac{1}{2\lambda_i} =: \alpha_0 < \infty, \quad t \in [0, T]$$

for all $T > 0$, where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. This implies

$$(4.2) \quad \mathbb{E} e^{\varepsilon_0 |M_t|^2} \leq C_0, \quad t \in [0, T]$$

for some constants $\varepsilon_0, C_0 > 0$ and all $T > 0$. Indeed, since

$$d|M_t|^2 = 2\langle M_t, dM_t \rangle + d\langle M \rangle_t,$$

by Itô's formula we have

$$\begin{aligned} d \exp \left[\frac{\varepsilon |M_t|^2 + 1}{\langle M \rangle_t + 1} \right] &= \exp \left[\frac{\varepsilon |M_t|^2 + 1}{\langle M \rangle_t + 1} \right] \frac{2\varepsilon}{\langle M \rangle_t + 1} \langle M_t, dM_t \rangle \\ &\quad - \exp \left[\frac{\varepsilon |M_t|^2 + 1}{\langle M \rangle_t + 1} \right] \left\{ \frac{\varepsilon |M_t|^2 + 1 - \varepsilon \langle M \rangle_t - \varepsilon - 2\varepsilon^2 |M_t|^2}{(\langle M \rangle_t + 1)^2} \right\} d\langle M \rangle_t. \end{aligned}$$

Since $\langle M \rangle_t \leq \alpha_0$, this process is a supermartingale when $\varepsilon \in (0, \frac{1}{2+\alpha_0}]$. So, (4.2) holds for $\varepsilon_0 = \frac{1}{(2+\alpha_0)^2}$ and some constant $C_0 > 0$. In particular, (4.2) implies

$$(4.3) \quad \mathbb{E} e^{\varepsilon_0 |\xi_T|^2} = \mathbb{E} e^{\varepsilon_0 M_T^2} \leq C_0 < \infty, \quad T > 0.$$

According to Theorem 2.1, Theorem 1.2 follows from the following four lemmas.

Lemma 4.1. *Assume (B1), (B2) and (B3). Then for any $t_0 > 0$ there exists a constant $c_0 > 0$ such that*

$$(P_{t_0} f)^2(\xi) \leq (P_{t_0} f^2(\eta)) e^{c_0 |\xi - \eta|^2}, \quad f \in \mathcal{B}_b(\mathbb{H}), \xi, \eta \in \mathbb{H} := \mathbb{H}_1 \times \mathbb{H}_2.$$

Proof. Let (X_t, Y_t) solve (1.7) with $(X_0, Y_0) = \eta$, and let (\bar{X}_t, \bar{Y}_t) solve the following equation for $(\bar{X}_0, \bar{Y}_0) = \xi$:

$$\begin{cases} d\bar{X}_t = (A\bar{X}_t + B\bar{Y}_t - L_1\bar{X}_t)dt, \\ d\bar{Y}_t = \left\{ Z(X_t, Y_t) - L_2\bar{Y}_t + \frac{1}{t_0} e^{-L_2 t} (Y_0 - \bar{Y}_0) + e^{-L_2 t} \frac{d}{dt} (t(t_0 - t) B^* e^{(t_0 - t) A^*} b) \right\} dt + \sigma dW_t, \end{cases}$$

where $b \in \mathbb{H}_1$ will be determined latter such that $(X_{t_0}, Y_{t_0}) = (\bar{X}_{t_0}, \bar{Y}_{t_0})$. We have

$$\begin{cases} d(X_t - \bar{X}_t) = \{A(X_t - \bar{X}_t) + B(Y_t - \bar{Y}_t) - L_1(X_t - \bar{X}_t)\}dt, \\ d(Y_t - \bar{Y}_t) = -\left\{L_2(Y_t - \bar{Y}_t) + \frac{1}{t_0}e^{-L_2t}(Y_0 - \bar{Y}_0) + e^{-L_2t}\frac{d}{dt}(t(t_0 - t)B^*e^{(t_0-t)A^*}b)\right\}dt. \end{cases}$$

Then

$$(4.4) \quad Y_t - \bar{Y}_t = \frac{t_0 - t}{t_0}e^{-L_2t}(Y_0 - \bar{Y}_0) - t(t_0 - t)e^{-L_2t}B^*e^{(t_0-t)A^*}b, \quad t \in [0, t_0],$$

and, since $BL_2 = L_1B$, $AL_1 = L_1A$,

$$\begin{aligned} (4.5) \quad X_t - \bar{X}_t &= e^{(A-L_1)t}(X_0 - \bar{X}_0) + \int_0^t \frac{t_0 - s}{t_0}e^{(A-L_1)(t-s)}Be^{-L_2s}(Y_0 - \bar{Y}_0)ds \\ &\quad - \int_0^t s(t_0 - s)e^{(A-L_1)(t-s)}Be^{-L_2s}B^*e^{A^*(t_0-s)}bds \\ &= e^{-tL_1}\left\{e^{At}(X_0 - \bar{X}_0) + \int_0^t \frac{t_0 - s}{t_0}e^{A(t-s)}B(Y_0 - \bar{Y}_0)ds \right. \\ &\quad \left. - \int_0^t s(t_0 - s)e^{A(t-s)}BB^*e^{A^*(t_0-s)}bds\right\}. \end{aligned}$$

According to (B3), the operator

$$\tilde{Q}_{t_0} := \int_0^{t_0} s(t_0 - s)e^{A(t_0-s)}BB^*e^{A^*(t_0-s)}ds$$

is invertible on \mathbb{H}_1 . So, letting

$$b = \tilde{Q}_{t_0}^{-1}\left\{e^{At_0}(X_0 - \bar{X}_0) + \int_0^{t_0} \frac{t_0 - s}{t_0}e^{A(t_0-s)}B(Y_0 - \bar{Y}_0)ds\right\},$$

we concluded from (4.4) and (4.5) that $(X_{t_0}, Y_{t_0}) = (\bar{X}_{t_0}, \bar{Y}_{t_0})$, and there exists a constant $C_1 > 0$ such that

$$(4.6) \quad |X_t - \bar{X}_t| + |Y_t - \bar{Y}_t| \leq C_1(|X_0 - \bar{X}_0| + |Y_0 - \bar{Y}_0|), \quad t \in [0, t_0].$$

Since A, B are bounded, σ is reversible, and Z is Lipschitz continuous, this implies that the process

$$\psi_t := \sigma^{-1}\left\{Z(X_t, Y_t) - Z(\bar{X}_t, \bar{Y}_t) + \frac{1}{t_0}e^{-L_2t}(Y_0 - \bar{Y}_0) + e^{-L_2t}\frac{d}{dt}(t(t_0 - t)B^*e^{(t_0-t)A^*}b)\right\}$$

satisfies

$$|\psi_t|^2 \leq C_2(|X_0 - \bar{X}_0|^2 + |Y_0 - \bar{Y}_0|^2), \quad t \in [0, t_0]$$

for some constant $C_2 > 0$. By the Girsanove theorem,

$$\tilde{W}_t := W_t + \int_0^t \psi_s ds, \quad t \in [0, t_0]$$

is the cylindrical Brownian motion on \mathbb{H}_2 under the probability measure $d\mathbb{Q} := R d\mathbb{P}$ for

$$R := \exp \left[- \int_0^{t_0} \langle \psi_s, dW_s \rangle - \frac{1}{2} \int_0^{t_0} |\psi_s|^2 ds \right],$$

and the equation for (\bar{X}_t, \bar{Y}_t) can be reformulated by

$$\begin{cases} d\bar{X}_t = (A\bar{X}_t + B\bar{Y}_t - L_1\bar{X}_t)dt, \\ d\bar{Y}_t = \{Z(\bar{X}_t, \bar{Y}_t) - L_2\bar{Y}_t\}dt + \sigma d\tilde{W}_t. \end{cases}$$

Then, by the weak uniqueness of the mild solutions to (1.7) and $(X_{t_0}, Y_{t_0}) = (\bar{X}_{t_0}, \bar{Y}_{t_0})$, we obtain

$$(P_{t_0}f(\xi))^2 = (\mathbb{E}_{\mathbb{Q}}f(\bar{X}_{t_0}, \bar{Y}_{t_0}))^2 = (\mathbb{E}[Rf(X_{t_0}, Y_{t_0})])^2 \leq (P_{t_0}f^2)(\eta)\mathbb{E}R^2 \leq (P_{t_0}f^2)(\eta)e^{c_0|\xi-\eta|^2}$$

for some constant $c_0 > 0$. \square

Lemma 4.2. *Assume (B1), (B2), (B3) and $\lambda := \lambda_1 - \lambda' > 0$. Let (X_t, Y_t) solve (4.1) for $X_0 = Y_0 = 0$. Then there exists $\varepsilon > 0$ such that $\sup_{t \geq 0} \mathbb{E}e^{\varepsilon(|X_t|^2 + |Y_t|^2)} < \infty$.*

Proof. By (B2), there exists a constant $c > 0$ such that

$$|Z(x, y)| \leq c + K_1|x| + K_2|y|, \quad x, y \in \mathbb{H}.$$

Combining this with (4.1), and noting that (B1) and (B3) imply $L_1 - A + \delta \geq \lambda_1$ and $L_2 \geq \lambda_1$, we obtain

$$(4.7) \quad \begin{aligned} |X_t| &\leq \int_0^t e^{-\lambda_1(t-s)} (\delta|X_s| + \|B\| \cdot |Y_s|) ds, \\ |Y_t| &\leq \int_0^t e^{-\lambda_1(t-s)} (c + K_1|X_s| + K_2|Y_s|) ds + |\xi_t|. \end{aligned}$$

Letting

$$\alpha = \frac{1}{2\|B\|} \left(\delta - K_2 + \sqrt{(K_2 - \delta)^2 + 4K_1\|B\|} \right),$$

by (B2) and (B3) we have $\alpha \in (0, \infty)$. Moreover, by the definition of α and (1.8) we have

$$\lambda'\alpha = \alpha\delta + K_1, \quad \alpha\|B\| + K_2 = \lambda'.$$

So,

$$(\alpha\delta + K_1)s + (\alpha\|B\| + K_2)t = \lambda'(\alpha s + t), \quad s, t \geq 0.$$

Combining this with (4.7) we obtain

$$\begin{aligned} \alpha|X_t| + |Y_t| &\leq \int_0^t e^{-\lambda_1(t-s)} \left\{ c + (\alpha\delta + K_1)|X_s| + (\alpha\|B\| + K_2)|Y_s| \right\} ds + |\xi_t| \\ &\leq \lambda' \int_0^t e^{-\lambda_1(t-s)} (\alpha|X_s| + |Y_s|) ds + |\xi_t| + \frac{c}{\lambda_1}. \end{aligned}$$

By Gronwall's inequality, this implies

$$(4.8) \quad \begin{aligned} \alpha|X_t| + |Y_t| &\leq |\xi_t| + \frac{c}{\lambda_1} + \lambda' \int_0^t e^{-\lambda(t-s)} \left(|\xi_s| + \frac{c}{\lambda_1} \right) ds \\ &\leq |\xi_t| + c_1 + \lambda' \int_0^t e^{-\lambda(t-s)} |\xi_s| ds, \quad t \geq 0 \end{aligned}$$

for some constant $c_1 > 0$ and $\lambda := \lambda_1 - \lambda' > 0$. Finally, by Jensen's inequality for the probability measure $\nu(ds) := \lambda e^{-\lambda(t-s)} ds$ on $(-\infty, t]$, we obtain

$$\begin{aligned} \exp \left[\varepsilon \left(\lambda' \int_0^t e^{-\lambda(t-s)} |\xi_s| ds \right)^2 \right] &= \exp \left[\frac{\varepsilon}{\lambda^2} \left(\lambda' \int_{-\infty}^t 1_{[0,t]}(s) |\xi_s| \nu(ds) \right)^2 \right] \\ &\leq \int_{-\infty}^t \exp \left[\frac{\varepsilon (\lambda')^2}{\lambda^2} 1_{[0,t]}(s) |\xi_s|^2 \right] \nu(ds) \\ &\leq c_2 + c_2 \int_0^t e^{-\lambda(t-s)} \exp [c_2 \varepsilon |\xi_s|^2] ds, \quad t, \varepsilon \geq 0 \end{aligned}$$

for some constant $c_2 > 0$. Combining this with (4.3) and (4.8), we finish the proof. \square

Lemma 4.3. *Assume (B1), (B2), (B3) and $\lambda := \lambda_1 - \lambda' > 0$. Then P_t has a unique invariant probability measure μ , and $\mu(e^{\varepsilon|\cdot|^2}) < \infty$ holds for some constant $\varepsilon > 0$.*

Proof. According to [20, Proposition 3.1], the Harnack inequality in Lemma 4.1 implies that P_t has at most one invariant probability measure. Let $(X_t, Y_t)_{t \geq 0}$ solve (1.7) for $X_0 = Y_0 = 0$. For every $t \geq 0$, let μ_t be the distribution of (X_t, Y_t) , which is a probability measure on $\mathbb{H} \times \mathbb{H}$. By the Markov property, if μ_t converges weakly to a probability measure μ as $t \rightarrow \infty$, then μ is an invariant probability measure of P_t , and, by Lemma 4.2 and Fatou's lemma, $\mu(e^{\varepsilon|\cdot|^2}) < \infty$ holds for some constant $\varepsilon > 0$. So, it remains to prove the weak convergence of μ_t as $t \rightarrow \infty$.

Consider the L^1 -Wasserstein distance

$$W(\nu_1, \nu_2) := \inf_{\pi \in \mathcal{C}(\nu_1, \nu_2)} \int_{\mathbb{H} \times \mathbb{H}} |\cdot| d\pi$$

for two probability measures ν_1 and ν_2 on $\mathbb{H} \times \mathbb{H}$, where $\mathcal{C}(\nu_1, \nu_2)$ is the set of all couplings of these two measures. If μ_t is a W -Cauchy family as $t \rightarrow \infty$, i.e.

$$(4.9) \quad \lim_{t_1, t_2 \rightarrow \infty} W(\mu_{t_1}, \mu_{t_2}) = 0,$$

then it converges weakly as $t \rightarrow \infty$, see e.g. [4, Theorem 5.4 and Theorem 5.6].

To prove (4.9), for any $t_2 > t_1 > 0$, let $(X_t, Y_t)_{t \geq 0}$ solve (1.7) for $X_0 = Y_0 = 0$, and let $(\tilde{X}_t, \tilde{Y}_t)_{t \geq t_2 - t_1}$ solve the following equation with $\tilde{X}_{t_2 - t_1} = \tilde{Y}_{t_2 - t_1} = 0$:

$$(4.10) \quad \begin{cases} d\tilde{X}_t = (A\tilde{X}_t + B\tilde{Y}_t - L_1\tilde{X}_t)dt, \\ d\tilde{Y}_t = \{Z(\tilde{X}_t, \tilde{Y}_t) - L_2\tilde{Y}_t\}dt + \sigma dW_t, \quad t \geq t_2 - t_1. \end{cases}$$

Then the distribution of (X_{t_2}, Y_{t_2}) is μ_{t_2} while that of $(\tilde{X}_{t_2}, \tilde{Y}_{t_2})$ is μ_{t_1} , so that

$$(4.11) \quad W(\mu_{t_1}, \mu_{t_2}) \leq \mathbb{E}(|X_{t_2} - \tilde{X}_{t_2}| + |Y_{t_2} - \tilde{Y}_{t_2}|).$$

It follows from (1.7), (4.1), (B2) and (B3) that for $t \geq t_2 - t_1$,

$$\begin{aligned} |X_t - \tilde{X}_t| &\leq e^{-\lambda_1(t-t_2+t_1)} |X_{t_2-t_1}| + \int_{t_2-t_1}^t e^{-\lambda_1(t-s)} (\delta |X_s - \tilde{X}_s| + \|B\| \cdot |Y_s - \tilde{Y}_s|) ds, \\ |Y_t - \tilde{Y}_t| &\leq e^{-\lambda_1(t-t_2+t_1)} |Y_{t_2-t_1}| + \int_{t_2-t_1}^t e^{-\lambda_1(t-s)} (K_1 |X_s - \tilde{X}_s| + K_2 |Y_s - \tilde{Y}_s|) ds. \end{aligned}$$

Then, for $\alpha, \lambda' > 0$ in the proof of Lemma 4.2,

$$\alpha |X_t - \tilde{X}_t| + |Y_t - \tilde{Y}_t| \leq e^{-\lambda_1(t+t_1-t_2)} (\alpha |X_{t_1}| + |Y_{t_1}|) + \lambda' \int_{t_2-t_1}^t e^{-\lambda_1(t-s)} (\alpha |X_s - \tilde{X}_s| + |Y_s - \tilde{Y}_s|) ds$$

holds for $t \geq t_2 - t_1$. By Gronwall's inequality we obtain

$$\alpha |X_{t_2} - \tilde{X}_{t_2}| + |Y_{t_2} - \tilde{Y}_{t_2}| \leq (\alpha |X_{t_1}| + |Y_{t_1}|) e^{-\lambda_1 t_1} \left(1 + \lambda' \int_{t_2-t_1}^{t_2} e^{\lambda'(t_2-s)} ds \right) = (\alpha |X_{t_1}| + |Y_{t_1}|) e^{-\lambda t_1}.$$

Noting that Lemma 4.3 implies $\sup_{t \geq 0} \mathbb{E}(|X_t| + |Y_t|) < \infty$, combining this with (4.11) we prove (4.9). Therefore, the proof is finished. \square

Lemma 4.4. *Assume (B1), (B2), (B3) and $\lambda := \lambda_1 - \lambda' > 0$. Then there exists a constant $C > 0$ such that for any two solutions (X_t, Y_t) and $(\tilde{X}_t, \tilde{Y}_t)$ to the equation (1.7),*

$$|X_t - \tilde{X}_t| + |Y_t - \tilde{Y}_t| \leq C(|X_0 - \tilde{X}_0| + |Y_0 - \tilde{Y}_0|) e^{-\lambda t}, \quad t \geq 0.$$

Proof. Similarly to the last part in the proof of Lemma 4.3,

$$\alpha |X_t - \tilde{X}_t| + |Y_t - \tilde{Y}_t| \leq e^{-\lambda_1 t} (\alpha |X_0 - \tilde{X}_0| + |Y_0 - \tilde{Y}_0|) + \lambda' \int_0^t e^{-\lambda_1(t-s)} (\alpha |X_s - \tilde{X}_s| + |Y_s - \tilde{Y}_s|) ds$$

holds for $t \geq 0$. Then by Gronwall's inequality,

$$\alpha |X_t - \tilde{X}_t| + |Y_t - \tilde{Y}_t| \leq e^{-\lambda t} (\alpha |X_0 - \tilde{X}_0| + |Y_0 - \tilde{Y}_0|), \quad t \geq 0.$$

This completes the proof. \square

5 Some Examples

We first present two examples to illustrate Theorem 1.1. When $A = 0, B = I$ and $b(y) = -\nabla V(y)$ for some $V \in C^2(\mathbb{R}^d)$, the first example reduces to the kinetic Fokker-Planck equation discussed in [14]; while in the second example m can be much larger than d for which the system becomes highly degenerate.

Example 5.1. Let σ and B be invertible, $m = d$, $Z(x, y) = b(y) - B^*x$ for some constant $\alpha > 0$. Assume

$$(5.1) \quad |b(y) - b(\bar{y})| \leq K|B(y - \bar{y})|, \quad \langle b(y) - b(\bar{y}), y - \bar{y} \rangle \leq -\beta|B(y - \bar{y})|, \quad y, \bar{y} \in \mathbb{R}^d$$

for some constants $K, \beta > 0$ and

$$(5.2) \quad \langle Ax, x \rangle \leq \gamma|B^*x|^2, \quad x \in \mathbb{R}^m$$

for some constant $\gamma \in \mathbb{R}$. If

$$(5.3) \quad \gamma < \frac{2\beta}{u + \sqrt{u^2 - 4 + 4\{(\|B\|\beta - 1)^+\}^2}},$$

for $u := 2 + (\|B^{-1}A^*\| + K)^2$, then assertions in Theorem 1.1 hold. In particular, it is the case for the kinetic Fokker-Planck equation where $A = 0$ and $b(y) = -y$.

Proof. Since B and σ are invertible and b is Lipschitz continuous, (A1) and (A2) hold. It remains to verify (A3). For any $r > 0$ we have

$$\begin{aligned} & \langle x - \bar{x} + rB(y - \bar{y}), A(x - \bar{x}) + B(y - \bar{y}) \rangle + \langle Z(x, y) - Z(\bar{x}, \bar{y}), y - \bar{y} + rB^*(x - \bar{x}) \rangle \\ & \leq \gamma|B^*(x - \bar{x})|^2 + r|B(y - \bar{y})|^2 + r(K + \|A(B^*)^{-1}\|)|B^*(x - \bar{x})| \cdot |B(y - \bar{y})| \\ & \quad - \beta|B(y - \bar{y})|^2 - r|B^*(x - \bar{x})|^2 \\ & = -(r - \gamma)|B^*(x - \bar{x})|^2 - (\beta - r)|B(y - \bar{y})|^2 + r(K + \|B^{-1}A^*\|)|B^*(x - \bar{x})| \cdot |B(y - \bar{y})| \\ & \leq -c(|B^*(x - \bar{x})|^2 + |B(y - \bar{y})|^2), \quad (x, \bar{x}), (y, \bar{y}) \in \mathbb{R}^{m+d} = \mathbb{R}^{2d} \end{aligned}$$

for some constant $c > 0$ provided

$$r(K + \|B^{-1}A^*\|) < 2\sqrt{(r - \gamma)^+(\beta - r)^+}.$$

So, (A3) holds provided

$$(5.4) \quad \sup_{r \in (0, \|B\|^{-1})} \frac{(r - \gamma)^+(\beta - r)^+}{r^2} > \frac{1}{4}(K + \|B^{-1}A^*\|)^2.$$

Below we show that (5.4) follows from (5.3) by considering two situations respectively.

(a) If $\gamma \leq 0$ then

$$\lim_{r \downarrow 0} \frac{(r - \gamma)^+(\beta - r)^+}{r^2} \geq \lim_{r \downarrow 0} \frac{\beta - r}{r} = \infty,$$

so that (5.4) holds.

(b) Let $\gamma > 0$. We first observe that (5.3) implies $\gamma < \beta \wedge \|B\|^{-1}$, since $u > 2$ so that

$$\frac{2\beta}{u + \sqrt{u^2 - 4 + 4\{(\|B\|\beta - 1)^+\}^2}} < \frac{2\beta}{2 + 2(\|B\|\beta - 1)^+} = \beta \wedge \|B\|^{-1}.$$

Take $r = \frac{2\gamma(\beta \wedge \|B\|^{-1})}{\gamma + \beta \wedge \|B\|^{-1}}$. We have $r \in (\gamma, \beta \wedge \|B\|^{-1})$ and (5.4) is implied by

$$\gamma^2\{\beta - 2(\beta \wedge \|B\|^{-1})\} + \gamma(\beta \wedge \|B\|^{-1})^2u - \beta(\beta \wedge \|B\|^{-1})^2 > 0.$$

It is trivial to see that this inequality follows from condition (5.3). \square

We note by making a linear transform $(\tilde{x}, \tilde{y}) = (Kx, KB y)$ for some invertible matrix K , Example 1.1 becomes to the following type of equations:

$$(5.5) \quad \begin{cases} d\tilde{X}_t = (\tilde{A}\tilde{X}_t + \tilde{Y}_t) dt, \\ d\tilde{Y}_t = \{\tilde{b}(\tilde{Y}_t) - \tilde{B}\tilde{X}_t\} dt + \tilde{\sigma} dW_t, \end{cases}$$

where $\tilde{A} := KAK^{-1}$, $\tilde{b}(y) := KBb((KB)^{-1}y)$, $\tilde{B} := KBB^*K^{-1}$ and $\tilde{\sigma} := KB\sigma$.

Example 5.2. Let σ be invertible, $m = kd$ for some natural number $k \geq 2$, and

$$\begin{aligned} By &= (0, \dots, 0, y) \in \mathbb{R}^{kd}, \quad y \in \mathbb{R}^d, \\ Z(x, y) &= b(y) - x_k, \quad y \in \mathbb{R}^d, x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^{kd}, \\ A(x_1, x_2, \dots, x_k) &= (\gamma x_2 - x_1, \gamma x_3 - x_2, \dots, \gamma x_k - x_{k-1}, 0), \quad x_1, \dots, x_k \in \mathbb{R}^d, \end{aligned}$$

where $\gamma \neq 0$ is a constant, and $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies (5.1). If

$$(5.6) \quad 0 < |\gamma| < 1 \wedge \frac{2\beta}{2 + K^2},$$

then assertions in Theorem 1.1 hold.

Proof. It is easy to see that when $\gamma \neq 0$, the rank condition in (A1) holds. Since b is Lipchitz continuous and σ is invertible, by Theorem 1.1 it suffices to prove (A3). For any $r > 0$ we have

$$\begin{aligned} &\langle x - \bar{x} + rB(y - \bar{y}), A(x - \bar{x}) + B(y - \bar{y}) \rangle + \langle Z(x, y) - Z(\bar{x}, \bar{y}), y - \bar{y} + rB^*(x - \bar{x}) \rangle \\ &= r|y - \bar{y}|^2 + \sum_{i=1}^{k-1} \left\{ \gamma \langle x_i - \bar{x}_i, x_{i+1} - \bar{x}_{i+1} \rangle - |x_i - \bar{x}_i|^2 \right\} \\ &\quad + \langle b(y) - b(\bar{y}), y - \bar{y} + r(x_k - \bar{x}_k) \rangle - r|x_k - \bar{x}_k|^2 \\ &\leq -(\beta - r)|y - \bar{y}|^2 - r|x_k - \bar{x}_k|^2 + rK|y - \bar{y}| \cdot |x_k - \bar{x}_k| \\ &\quad - \sum_{i=1}^{k-1} \left\{ |x_i - \bar{x}_i|^2 - \frac{|\gamma|}{2}|x_i - \bar{x}_i|^2 - \frac{|\gamma|}{2}|x_{i+1} - \bar{x}_{i+1}|^2 \right\} \\ &\leq -\sum_{i=1}^{k-1} (1 - |\gamma|)|x_i - \bar{x}_i|^2 - \left(r - \frac{|\gamma|}{2} - \frac{rK^2}{4\alpha} \right) |x_k - \bar{x}_k|^2 - (\beta - r - \alpha r)|y - \bar{y}|^2, \quad \alpha > 0. \end{aligned}$$

So, (A3) holds provided $|\gamma| < 1$ and

$$\sup_{r \in (0, 1 \wedge \frac{\beta}{1+\alpha}), \alpha > 0} \left(r - \frac{|\gamma|}{2} - \frac{K^2 r}{4\alpha} \right) > 0.$$

Letting $r \uparrow 1 \wedge \frac{\beta}{1+\alpha}$, we conclude that (A3) holds provided $|\gamma| < 1$ and

$$\sup_{\alpha > 0} \left(1 \wedge \frac{\beta}{1+\alpha} \right) \left(1 - \frac{K^2}{4\alpha} \right) > \frac{|\gamma|}{2}.$$

By taking $\alpha = \frac{1}{2}K^2$ we see that this inequality follows from (5.6). \square

Now, we consider Theorem 1.2. It is easy to illustrate this result for $\mathbb{H}_1 = \mathbb{H}_2$, $L_1 = L_2$, $B = I$ (the identity) and $\alpha = \|A\|$. Below we present an example in the spirit of Example 5.2 that \mathbb{H}_2 is a subspace of \mathbb{H}_1 .

Example 5.3. Let $\{u_i\}_{i \geq 1}$ be an orthonormal basis on \mathbb{H}_1 , and let $\mathbb{H}_2 = \overline{\text{span}}\{u_{2i} : i \geq 1\}$. Next, take $B = I_{\mathbb{H}_2}$ and

$$L_1 u_{2i} = \lambda_i u_{2i}, \quad L_1 u_{2i-1} = \lambda_i u_{2i-1}, \quad i \geq 1$$

for $0 < \lambda_i \uparrow \infty$ with $\sum_{i \geq 1} \lambda_i^{-1} < \infty$. Moreover, let $L_2 = L_1|_{\mathbb{H}_2}$ and

$$Ax = \gamma \lambda_1 \sum_{i=1}^{\infty} \langle x, u_{2i} \rangle u_{2i-1}, \quad x \in \mathbb{H}_1$$

for some constant $\gamma \in \mathbb{R}$. Finally, let Z satisfy

$$|Z(x, y) - Z(\bar{x}, \bar{y})| \leq \alpha \lambda_1 |x - \bar{x}| + \beta \lambda_1 |y - \bar{y}|$$

for some constants $\alpha, \beta \geq 0$. Then all assertions in Theorem 1.1 provided

$$(5.7) \quad \sqrt{1 + \gamma^2} + 4\beta + \sqrt{(2\beta - 1 - \sqrt{1 + \gamma^2})^2 + 8\alpha} < 7.$$

Proof. It is easy to see that $BL_2 = L_1B$, $AL_1 = L_1A$. According to Theorem 1.2, it suffices to prove

(a) For some $\delta > 0$ such that $L_1 - A \geq \lambda_1 - \delta$ and the condition (1.8) hold.

(b) For any $t_0 > 0$, Q_{t_0} is invertible on \mathbb{H}_1 .

Proof of (a). We have

$$\begin{aligned} \langle (L_1 - A)x, x \rangle &= \langle L_2 \pi x, \pi x \rangle - \langle Ax, x \rangle \\ &\geq \lambda_1 \sum_{i \geq 1} \langle x, u_{2i} \rangle^2 - \gamma \sum_{i \geq 1} \langle x, u_{2i} \rangle \langle x, u_{2i-1} \rangle \\ &\geq (\lambda_1 - \delta) \sum_{i \geq 1} \langle x, u_{2i} \rangle^2 - \frac{\gamma^2}{4\delta} \sum_{i \geq 1} \langle x, u_{2i-1} \rangle^2, \quad x \in \mathbb{H}_1. \end{aligned}$$

Setting $\frac{\gamma^2}{4\delta} = \delta - \lambda_1$ we obtain

$$\delta = \frac{1 + \sqrt{1 + \gamma^2}}{2} \lambda_1,$$

for which $L_1 - A \geq \lambda_1 - \delta$ holds as required, and the condition (5.7) is equivalent to (1.8).

Proof of (b). We may simply assume $\gamma \lambda_1 = 1$, so that

$$A^*x = \sum_{i=1}^{\infty} \langle x, u_{2i-1} \rangle u_{2i}, \quad x \in \mathbb{H}_1.$$

Since $A^2 = (A^*)^2 = 0$ and BB^* is the orthogonal projection onto \mathbb{H}_2 , for any $x \in \mathbb{H}_1$ we have

$$\begin{aligned} e^{sA}BB^*e^{sA^*}x &= (I + sA)BB^*\{x + sA^*x\} \\ &= \sum_{i=1}^{\infty} (\langle x, u_{2i} \rangle + s\langle x, u_{2i-1} \rangle) \{u_{2i} + su_{2i-1}\}. \end{aligned}$$

Then

$$\begin{aligned} \langle Q_{t_0}x, x \rangle &= \sum_{i=1}^{\infty} \int_0^{t_0} \{ \langle x, u_{2i} \rangle^2 + 2s\langle x, u_{2i-1} \rangle \langle x, u_{2i} \rangle + s^2 \langle x, u_{2i-1} \rangle^2 \} ds \\ &= t_0 \sum_{i=1}^{\infty} \left\{ \langle x, u_{2i} \rangle^2 + t_0 \langle x, u_{2i-1} \rangle \langle x, u_{2i} \rangle + \frac{t_0^2}{3} \langle x, u_{2i-1} \rangle^2 \right\} \\ &\geq t_0 \sum_{i=1}^{\infty} \left\{ (1-r) \langle x, u_{2i} \rangle^2 + \left(\frac{1}{3} - \frac{1}{4r} \right) t_0^2 \langle x, u_{2i-1} \rangle^2 \right\}, \quad r > 0. \end{aligned}$$

Taking $r \in (0, 1)$ but close enough to 1, we conclude that $\langle Q_{t_0}x, x \rangle \geq c|x|^2$ holds for some constant $c > 0$ and all $x \in \mathbb{H}_1$. Thus, Q_{t_0} is invertible. \square

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